I saw the figure 5 in gold
Charles Demuth, 1928

Mathematics

Among the rain
and lights
I saw the figure 5
in gold
on a red
fire truck
moving
tense
unheeded
to gong clangs
siren howls
and wheels rumbling
through the dark city

The Great Figure
William Carlos Williams, 1921

This session will consider the history of mathematics. Numbers are the basis of much of scientific thinking. Numbers – particularly prime numbers - have often been imbued with magical or mystical qualities. There are three persons in the trinity, five regular polyhedrons (Platonic solids), seven wonders of the world, etc.

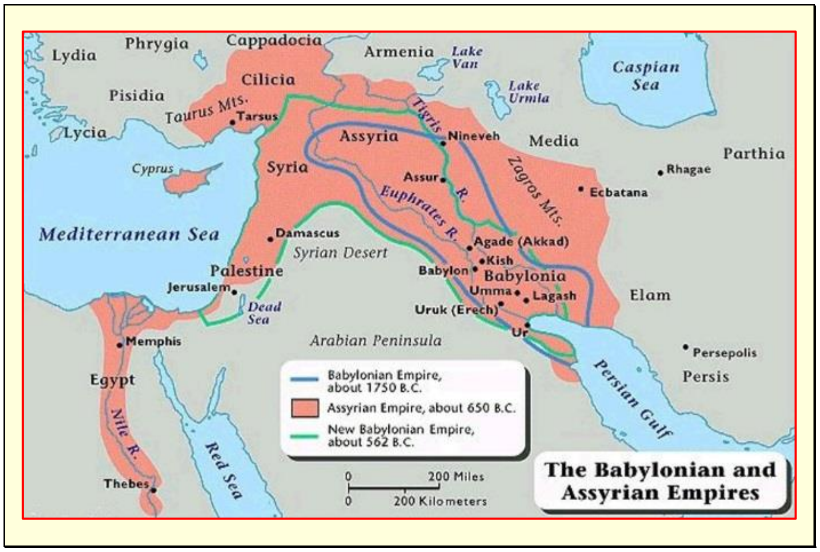
The painting by Charles Demuth depicts the number 5. It derives from a poem by William Carlos Williams. The poem came first, the painting followed. *The Great Figure* is one of the few poems about which a painting has been made. Usually *ekphrasis* (art about art) goes the other way – poets describe paintings. Painters may illustrate episodes from epic poems but painters do not generally represent whole poems.

Williams (1883-1963) was a poet and pediatrician. He became one of the most important of the American modernist poets, writing in free verse that was clear and idiomatic.In his *Autobiography*, Williams wrote:
Once on a hot July day coming back exhausted from the Post Graduate Clinic, I dropped in as I sometimes did at Marsden's studio on Fifteenth Street for a talk, a little drink maybe and to see what he was doing. As I approached his number I heard a great clatter of bells and the roar of a fire engine passing the end of the street down Ninth Avenue. I turned just in time to see a golden figure 5 on a red background flash by. The impression was so sudden and forceful that I took a piece of paper out of my pocket and wrote a short poem about it.

A deeper meaning can perhaps be elicited from the title, which could refer to a famous person. Thus the famous person is carried around the dark city to the accolades of gongs and sirens. The New York Fire Department still has Engine 5 stationed at 14th St and 1st Ave. The squadron there was involved in the rescue effort at the World Trade Center.

The recitation of the poem is by James Carew. It is available on
https://youtube.com/watch?v=BpN8YYdTIrw

This presentation will be concerned mainly with the history of mathematics. However, we shall occasionally digress and mention some stories about logic and the scientific method.



This map shows some of the ancient empires that were centered near the Tigris and Euphrates valleys. The ancient Sumerians were the first "civilization" in this region beginning around 5000 BCE. The Akkadian Empire (with its capital at the lost city of Akkad) lasted for about two centuries 2300-2100 BCE and was then superseded by the Babylonian Empire which lasted from 1900 to 539 BCE. During part of this time the empire was controlled by the Assyrians (911-609 BCE). The Assyrians conquered Northern Israel but not the southern part.

The Sumerians were the first people to use writing with the cuneiform inscriptions surviving from around 3000 BCE. Much of the early inscriptions appear to represent accounting for grain deposits and trading. Before we look at these writings, we need to understand a little bit about number systems. Nowadays we count in tens. In those early days they count in 60s.

Number Systems

The number **245** in the decimal (base 10) system means

2 X 10^2 or 200 + **4** X 10^1 or 40 + **5** X 10^0 or 5

The number **1010** in the binary (base 2) system means

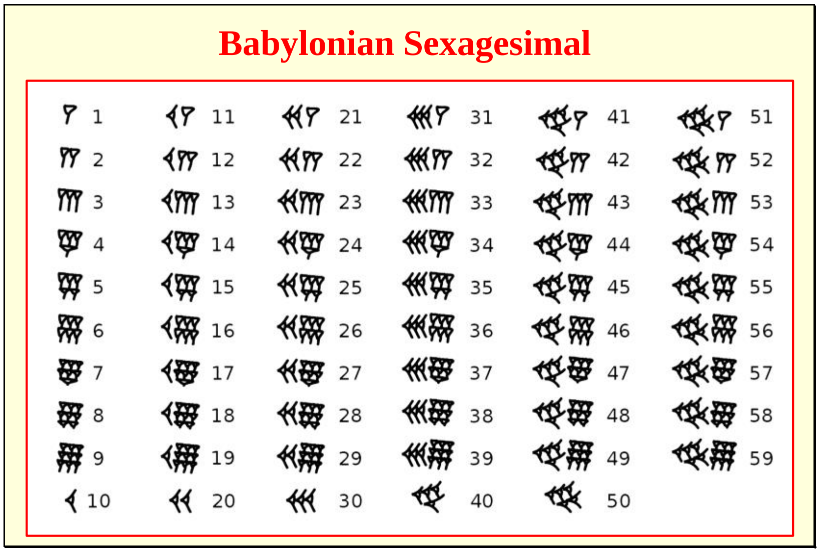
1 X 2^3 or 8 + **0** + **1** X 2^1 or 2 + **0**

Binary	**0**	**1**	**10**	**11**	**100**	**101**	**110**	**111**	**1000**	**1001**	**1010**	**1011**	**1100**	**1111**
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15
Decimal	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Hex	0	1	2	3	4	5	6	7	8	9	A	B	C	D

There are 10 types of people – those that understand the binary number system and those that do not.

Binary is the system used in computers. There are only two values; 0 and 1. This fits with the transistor circuits being off or on. However, the system is cumbersome. Many computers use either an octal system – based on 8 – or a "hexadecimal" system - based on 16.

In the decimal system, the numbers after the decimal represent the numbers divided by (rather than multiplied by) powers of ten:
1.25 means 1 X 100 or 1 + 2 /10 or 0.2 + 5/(10)2 or 0.05



The Sumerians, and the Babylonians that followed them, used a number-system based on 60 ("sexagesimal"). Most numbering is done on the basis of 10 ("decimal"), probably because we first began to count on using our eight fingers and two thumbs (out "digits"). In the decimal system the written numerals denote 1-9 and the position in a multi-digit number denotes the power of ten. Thus

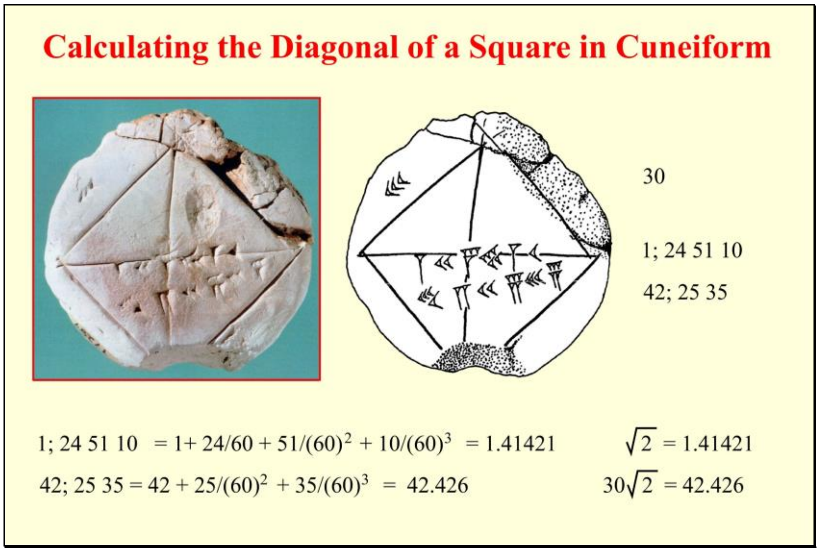
243.9 means 2 times 100 plus 4 times 10 plus 3 times 1 plus 9 times 1/10.

Sexagesimal has advantages over decimal since it can be divided without fractions by 3 and 4. The system persists in our measurements of time: the 60 seconds per minute and 60 minutes per hour. The duodecimal system of 12 pence per shilling has similar advantages since it can be divided by 3 and 4.

Babylonian sexagesimal was actually a hybrid of sexagesimal and decimal. There were not 60 separate numerals. Rather there are nine simple numerals from 1 to 10 and five numerals to represent multiples of 10 from 10 to 60. Most of the numerals therefore are represented by two symbols.

The modern way of writing sexagesimal uses a colon or semicolon to represent the place location between integer and fraction and spaces between the double symbols. The colon is similar to the way we represent time, e. g. 4:15 pm

e.g. 5: 15 45 in sexagesimal
represents 5 + 15/60 + 45/3600
which is 5 + 0.25 + 0.0125 or 5.2625 in decimal



This is a clay tablet YBC 7289 in the Yale Babylonian Collection. It dates back to about 1700 BCE. It likely represents a student's exercise to calculate the length of a diagonal of a square with a side of 30.

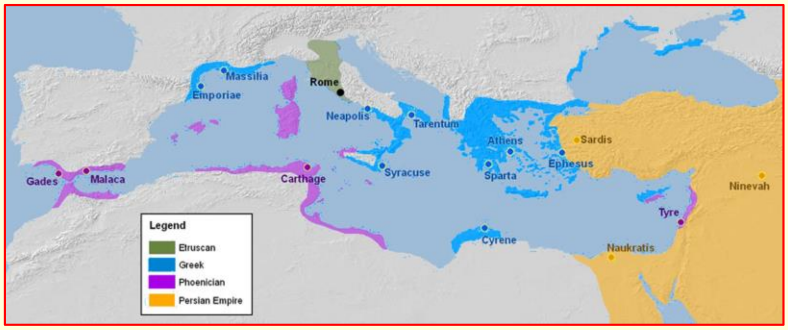
Because positional notation (that which separates the integer part from the fractional part of a number, i.e. decimal point) is ambiguous in Babylonian arithmetic, other interpretations are possible. See
https://en.wikipedia.org/wiki/YBC_7289

Other Babylonian tablets contain tables with values of the diagonal for rectangles with sides various integer sizes.

These diagonals can be calculated using the principle of the Pythagorean Theorem. The square on the hypotenuse of a right angle triangle is the sum of the squares on the other two sides. In terms of a rectangle the square on the diagonal is equal to the sum of the square on the two different sides. Whether they were aware of the theorem or just measured these values is not known. Nevertheless these values are very accurate.

One must also be impressed by how these calculations were carried out using integers only. They had no way or representing a fractional value other than as a ratio of two integers. This made arithmetic very cumbersome compared to arithmetic in the decimal system.

Using fractions 1.25 + 1.3333 would be represented as 5/4 + 4/3. The numbers could only be added by finding a common denominator (12): 15/12 +16/12 or 31/12.

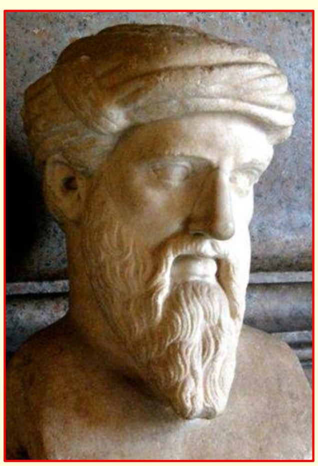


Mediterranean during the Iron Age (~ 500 BCE)

The peoples of Ancient Greece consisted of the Ionians on the Western coast of modern Turkey and in the islands of the Eastern Aegean, the Delian League of city states centered around Athens, the Peloponnesus (Sparta) and Greek colonies in various regions of the Mediterranean. The most important colonies were in Magna Graecia in Southern Italy and Sicily.

Alexander the Great (356-323 BCE) invaded and conquered the Persian Empire. Alexander's Empire at its height in 320 BCE stretched eastward to the northern part of India and southward to Egypt. The city of Alexandria was founded in 332 BCE

Pythagoras was born in Samos, an island just off the coast of Turkey near Ephesus. At the age of 40 he went to Croton a colony in Magna Graecia (located on the Southern coast of Calabria).



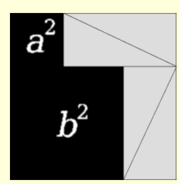
Roman copy of a bust of Pythagoras from ~500 BCE

Pythagoras (570-495 BCE)

Among the ideas attributed to Pythagoras:

(i) the claim that the Earth is a sphere
(ii) the harmony between notes with frequencies separated by intervals of 3:2 (perfect fifth) and 2:1 (octaves)
(iii) the theorem about right-angled triangles

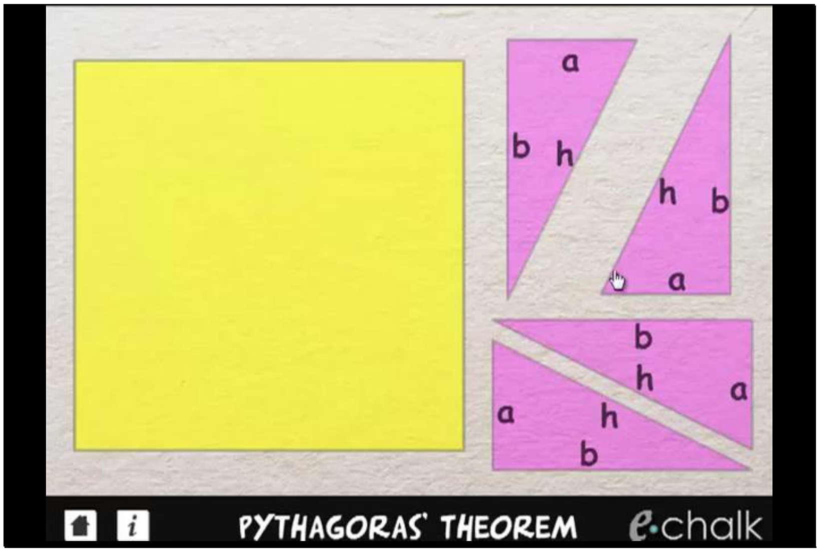
In a right-angled triangle, the square on the hypotenuse (c) equals the sum of the squares on the other two sides (a and b).



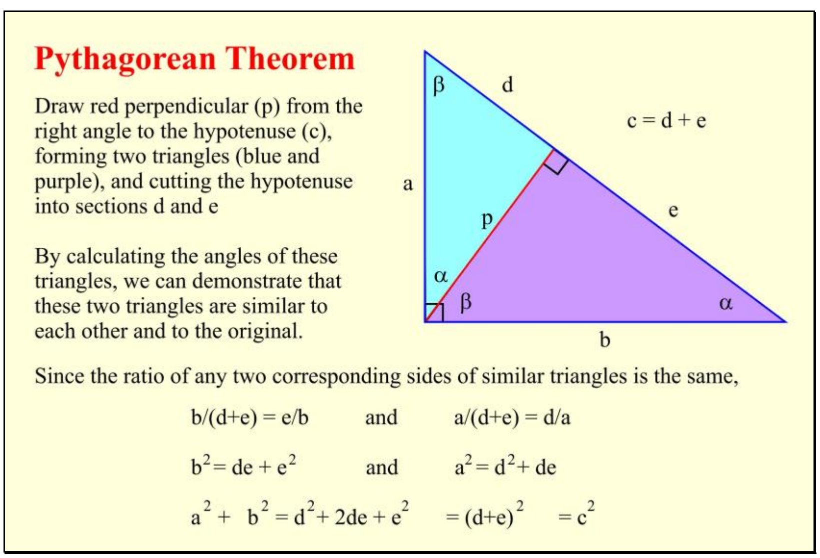
These attributions are uncertain. They may have originated with other Ionian thinkers or have come from knowledge transmitted from the East (India and Mesopotamia) through the Persian Empire.

Depending on how the words are translated, the *Baudhayana Sutra* which was likely written about 700 BCE appears to state the Pythagorean Theorem, thus providing the earliest known written version of the theorem. Whether the Vedic peoples in India originated the theorem or whether it came to them from the Babylonians is unknown.

The gif illustration represents one way of visually demonstrating the Pythagorean Theorem



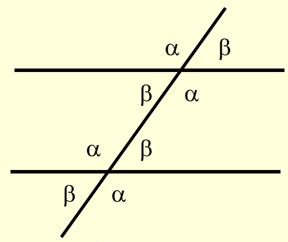
This is supposedly Pythagoras' geometric proof of his theorem
The video comes from echalk via
https://youtube.com/watch?v=T2K11eFepcs



The simplest proof of the Pythagorean Theorem involves drawing a perpendicular from the right-angled corner to the hypotenuse. This forms two triangles. Knowing that the angles of a triangle sum to 180 degrees, we can calculate the angles of these two triangles and prove that they are both “similar” to the original triangle (the angles are the same and the ratios between the sides are the same). Then we use algebra to derive the theorem from the ratios of the sides

Euclid (~325-285 BCE)

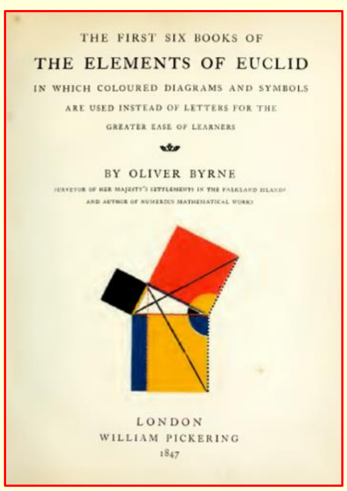
Euclid as portrayed by Raphael, *School of Athens*, Vatican, 1509



Definition 23: Parallel straight lines are lines which, being in the same plane and being extended indefinitely in both directions never meet.

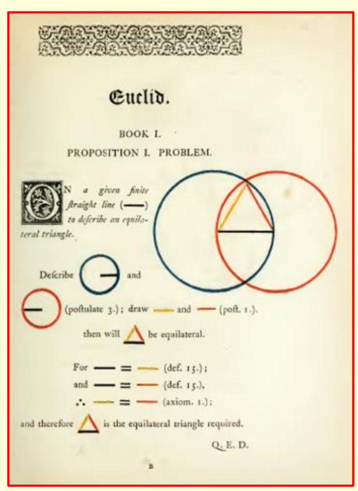
Proposition 27: If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Euclid worked in Alexandria. There is no known portrait of Euclid. The illustration shows a detail from a Raphael fresco in the Vatican which depicts many of the Greek philosophers and scientists. Raphael supposedly used the Donato Bramante, the architect who designed much of the Vatican, as a model for Euclid.


THE FIRST SIX BOOKS OF
THE ELEMENTS OF EUCLID
IN WHICH COLOURED DIAGRAMS AND SYMBOLS ARE USED INSTEAD OF LETTERS FOR THE GREATER EASE OF LEARNERS

BY OLIVER BYRNE

LONDON
WILLIAM PICKERING
1847


Euclid.

BOOK I.
PROPOSITION I. PROBLEM.

On a given finite straight line (—) to describe an equilateral triangle.

Describe and (postulate 3.); draw and (post. 1.).

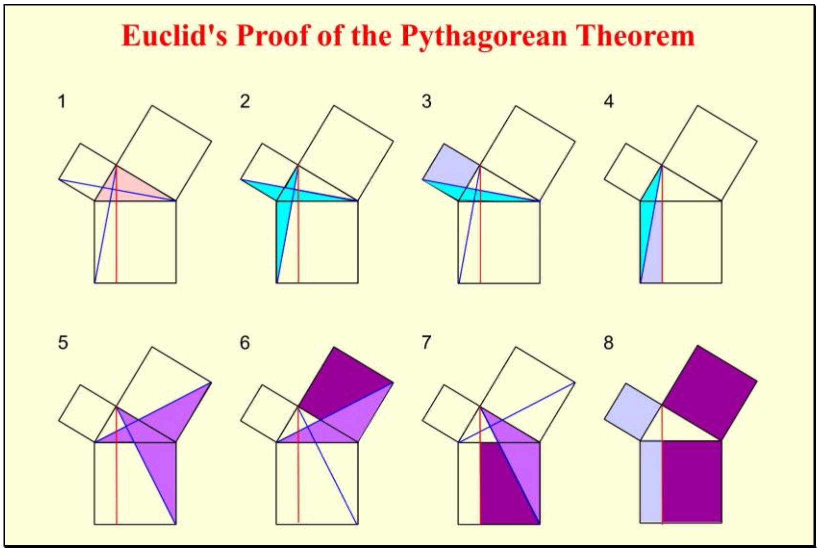
then will be equilateral.

For = (def. 15.);
and = (def. 15.),
∴ = (axiom. 1.);
and therefore is the equilateral triangle required.

Q. E. D.

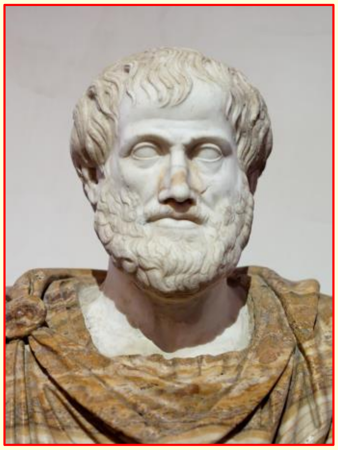
Euclid’s *Elements* continues to be taught today. This illustration shows an edition of the first 6 books of the *Elements* published in 1847. Oliver Byrne used color to indicate the lines, angles and shapes. For some this is much simpler to understand than the usual designation with letters: lines as AB, angles as ∠ABC and triangles as ΔABC

In Proposition 1, the proof refers to definition 15 (of a circle, which states that the circumference is at all times equidistant to the center) and to axiom 1 (things equal to the same are equal to each other)



This is Euclid's demonstration of the Pythagorean Theorem using the style of Oliver Byrne. This comes from *Elements* I.47

- Construct the squares on the sides of the right-angled triangle (pink). Draw red line from the right-angle corner of the triangle parallel to the sides of the square on the hypotenuse. Draw blue lines from the corners of the two squares on the left to the opposite corner of the triangle.
- The two triangles indicated in light blue are congruent because their smaller sides are equal and because the angles between these sides equals a right angle plus the larger of the two non right angles in the original pink triangle.
- Because the grey square and the upper light blue triangle are located between the parallel lines the square's area is 2 times the area of the triangle.
- Because the grey rectangle and the other light blue triangle are located between the parallel lines the rectangle's area is 2 times the area of the triangle. Therefore the area of the grey square is equal to the area of the grey rectangle.
- Steps 5-7 are the similar to 2-4 and prove that the dark purple square is equal in area to the dark purple rectangle. Thus the square on the hypotenuse is equal to sum of the squares on the other two sides.



Roman copy of a portrait bust by Lysippos (~350 CE)

Aristotle (384–322 BC)

Born in Northern Greece, Aristotle studied with Plato in Athens. He wrote extensively in all areas of science and philosophy. His work became the foundation of Western Thought. During the European Middle Ages his writings were considered as having the same authority as scripture. Although he was aware of inductive logic – arguing from the particular to the general (or from the known to the unknown) – and used it in his scientific works, he contributed mainly to deductive logic – what can be concluded from what is already known.

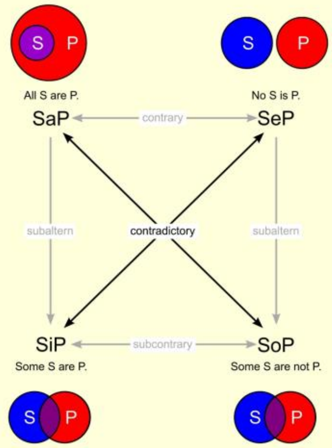
Aristotle had significant effects on history as well as philosophy – he taught the young Alexander the Great.

Syllogistic Logic

The Death of Socrates, 1787
Jacques-Louis David

Major Premise:	All men are mortal
Minor Premise:	Socrates is a man
Conclusion:	Socrates is mortal

The Square of Opposition



Medieval logicians exhaustively analyzed the types of propositions (often called **a e i** and **o**) and the validity of the conclusions. For example syllogisms formed as **aaa** (the Socrates syllogism) or **eae** (No reptiles have fur; snakes are reptiles; no snakes have fur) are valid. The logicians named the valid syllogisms using mnemonics such as B**a**rb**ara** and C**esare,** and diagrams such as the square of opposition. Much thought was given to logic but no new knowledge ("science") was derived.

The propositions are illustrated with Euler diagrams, invented by the German mathematician Leonhard Euler (1707-1783) and later modified by the British logician John Venn (1834-1923).

Francis Bacon (1561-1626)

The Scientific Method

The ancients paid much more attention to deductive logic than inductive reasoning. They did science, but they did not think much about how they did it or how they proved something was true. During the early 17th Century Francis Bacon proposed that knowledge should be obtained by extensive observation and experiment. Laws governing the world could then be inferred from the empirical data. This was a major step but it was not until the 19th century that the concept of making and testing hypotheses and the idea of falsifiability came about.

Portrait is a copy of an original (from ~1618) made 120 years later.

At the same time as the Catholic Church was promoting papal infallibility (1869-1870), the American philosopher Charles S. Peirce was describing how science was based on fallibilism. A scientific statement is one that can be proven false by observation or experiment. We accept as true statements about the world that can be tested and have not yet been refuted by such tests.

According to Peirce, all human knowledge is uncertain:

> Fallibilism is the doctrine that our knowledge is never absolute but always swims, as it were, in a continuum of uncertainty and of indeterminacy (1897).

Karl Popper extended this idea to state that science is composed of falsifiable statements that have not yet been falsified when tested:

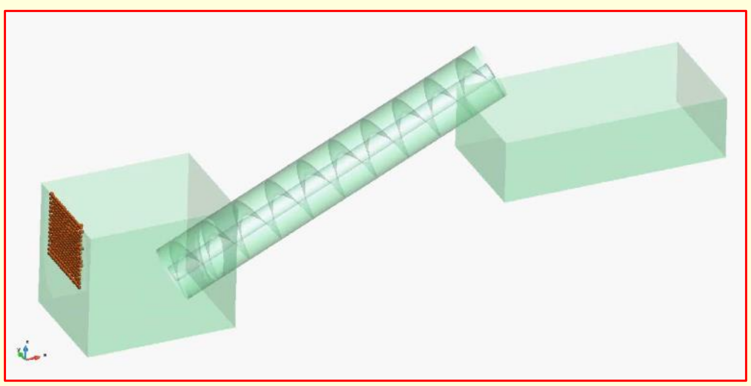
> It must be possible for an empirical scientific system to be refuted by experience (1959)

Archimedes (287-212 BCE)

Archimedes was born in Syracuse in Magna Graecia. He was killed during the Roman invasion of Southern Italy as part of the Second Punic War.

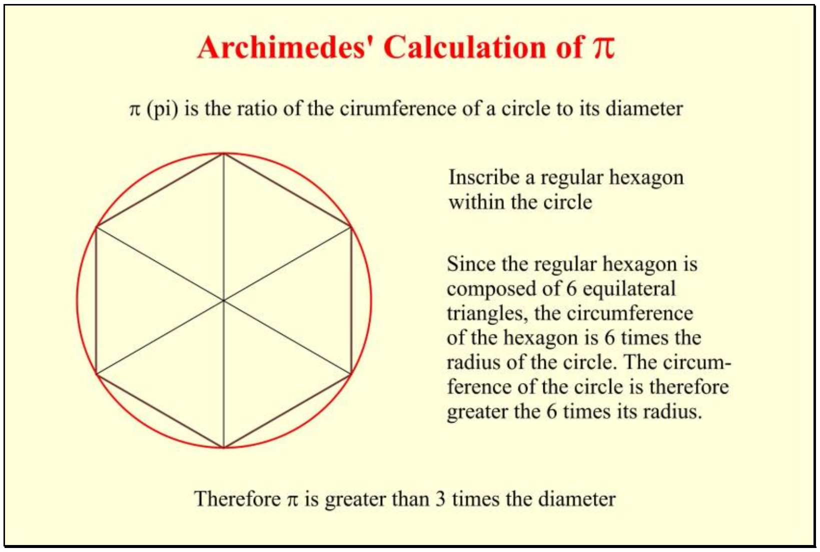
Given the problem of determining whether the king's crown was pure gold or adulterated with silver, Archimedes needed to determine the density of the crown (higher if it was pure gold). For this he needed to know the volume of the irregularly shaped crown. In his bath he realized that this was equal to the volume of water that the crown displaced. He then supposedly ran naked through the streets of Syracuse shouting *Eureka!* ("I have found it!")

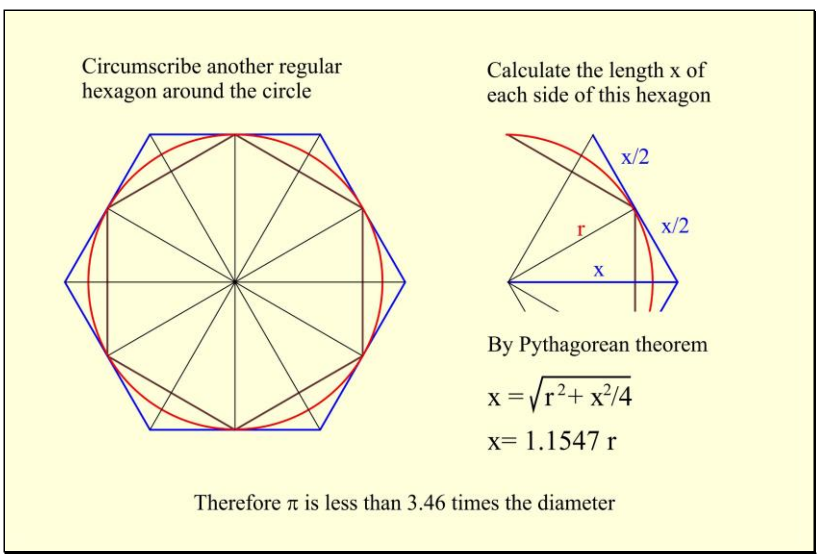
The story of the Eureka moment was first told by the Roman Vitruvius in the 1st Century CE. It is likely apocryphal. This is discussed by
https://www.scientificamerican.com/article/fact-or-fiction-archimede/



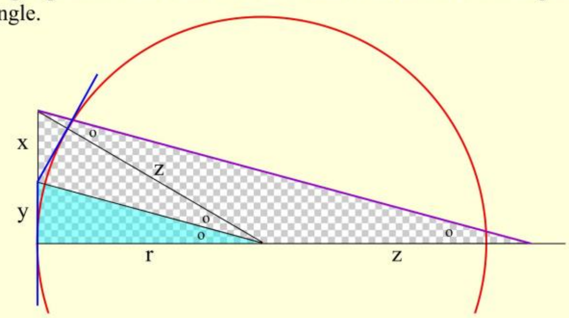
This design of a machine to raise water from one level to another, e.g. from a river to an irrigation canal is attributed to Archimedes
https://www.youtube.com/watch?v=0PgA6Dz7f_M

Archimedes also proposed a way to calculate π:



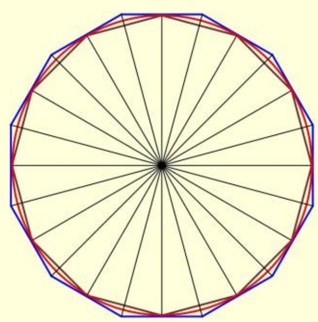


Euclid: If a straight line bisects an angle of a triangle and cuts the base then the resulting segments of the base have the same ratio as the remaining sides of the triangle.



Proof involves drawing the purple line parallel to the bisector. This forms an isosceles triangle with z-sides equal. Then the checkered triangle is similar to the superimposed smaller cyan triangle. This entails $(x+y)/(r+z) = y/r$. We can calculate z from the Pythagorean theorem because of the right-angled triangle with sides x+y, r and z. Ultimately we can get y in terms of x+y and r.

Inscribe and circumscribe regular dodecagons around the circle



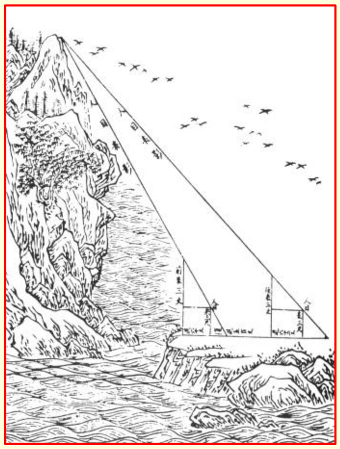
Repeat with 24, 48 and 96 sided polygons

π

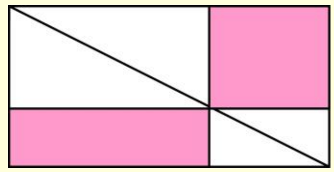
Sides	Lower	Upper
6	3.0000	3.4641
12	3.1058	3.2154
24	3.1326	3.1597
48	3.1394	3.1461
96	3.1410	3.1427

3.141592653

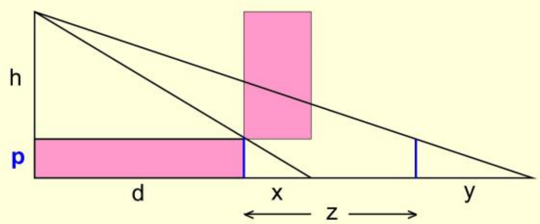
Chinese Surveying



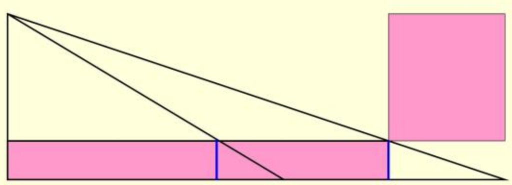
Chinese geometry was as advanced as that of Euclid. However, during the reign of Qin Shi Huang (221-210 BCE) all the books were burned. In 179 CE this knowledge was re-collected in *Nine Chapters on the Mathematical Art.* In 263 CE Liu Hui edited this work and added a commentary. One of his sample problems was to calculate the height of a sea island. To do this he used the theorem of the inscribed triangles:



The theorem states that the pink rectangles have equal area. This can be derived from the fact that the diagonal separates the outside rectangle to two equal halves. It also separates the smaller white rectangles into equal halves.



$$pd = xh$$



$$p(d + z) = yh$$

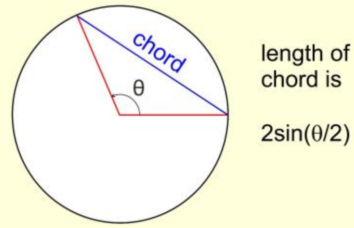
$$pd + pz = yh$$

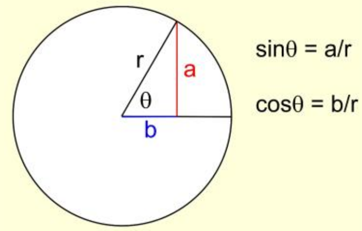
$$xh + pz = yh$$

$$h = pz / (y - x)$$

Ancient Trigonometry

The Ionian Greek astronomer Hipparchus (190-120 BCE) provided the first table of chord lengths. These were then included in the *Almagest* (*al + megiste*, "greatest") of Claudius Ptolemy (100-170 CE) in Alexandria. The actual convention of *sine* and *cosine* derives from the Indian astronomer Aryabhata (476-550 CE). Some of these ideas are in the earlier Sanskrit *Surya Siddhanta*. The trigonometric values became fully tabulated by Islamic mathematicians in the 10th Century CE. These were then used by European astronomers such as Copernicus and Kepler.





The *Surya Siddhanta* dates back to the 6th Century BCE although it was revised many times thereafter. It is unknown how much of the trigonometry in the Sanskrit texts came from the original writings and how much may have been added later when Greek ideas were transmitted to the East through the Alexandrian Empire.

Ancient Algebra

The Compendious Book on Calculation by Completion and Balancing,
Al Khwarizmi, 820 CE

The father of algebra is generally considered to be the Alexandrian Diophantus (210-294 CE) who introduced the idea of using a symbol such as x to represent an unknown quantity. However, he was limited by not recognizing negative numbers or calculating in fractions. The Indian Brahmagupta (598-668 CE) provided the first explicit solution of the quadratic equation:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Persian polymath Al Khwarizmi (780-850 CE) was the first to systematize the solution of algebraic equations and to invent the name algebra ("reunion of broken parts").

The name Al Khwarizmi is the source of our modern word "algorithm."

Another contributor to the development of Algebra was the Persian Omar Khayyam (1048-1131 CE), who is now more famous for his poetry written in quatrains (*rubaiyat*). He first described the coefficients of the binomial expansion (Pascal's triangle):

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
```

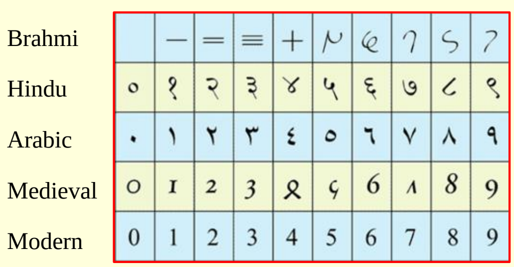
This gives the coefficients for $(a+b)^n$

From the *Rubaiyat* translated by Edward FitzGerald (1889):

> The Moving Finger writes; and, having writ,
> Moves on: nor all thy Piety nor Wit
> Shall lure it back to cancel half a Line,
> Nor all thy Tears wash out a Word of it.

Modern Numbers

The modern way of writing numbers uses
- (i) the decimal system
- (ii) a positional indicator (decimal point)
- (iii) positive and negative numbers
- (iv) zero as a placeholder between positive and negative



Our numerals derive from the Hindu-Arabic system first proposed by the Indian mathematician Aryabhata in 499 CE, and systematized by the Persian Al Khwarizmi in his book *On the Calculation with Hindu Numerals* (825 CE).

This and other facts about the history of the numeral system is available at http://archimedes-lab.org/numeral.html

The first recorded symbol for zero is in the Bakhshali manuscript found in modern Pakistan. This may date back to 200 CE.

As we became able to express numbers in this way we could begin to consider the properties of various kinds of numbers. This is the beginning of number theory.

Prime Numbers

The Sieve of Eratosthenes (276-194 BCE) eliminates multiple so each prime number beginning with 2 as non-prime numbers. The next non-eliminated number is prime.

Eratosthenes was born in Cyrene in Libya. He became the chief librarian of Athens. Eratosthenes was also famous for estimating the diameter of the Earth.

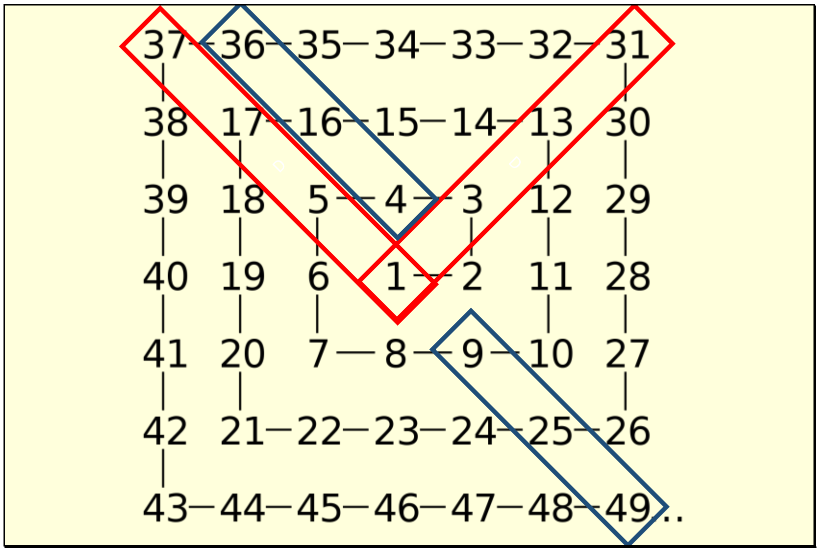
A prime is a number that cannot be formed by formed by multiplying together two or more smaller integers. A prime number is only divisible by itself and one. Non-primes are also called composite numbers. These can be expressed as the product of their constituents.
Various techniques more efficient than Eratosthenes sieve have been used to determine primes. However, a formula which would identify all primes has not yet been found.

Prime numbers are one of many types of numbers:

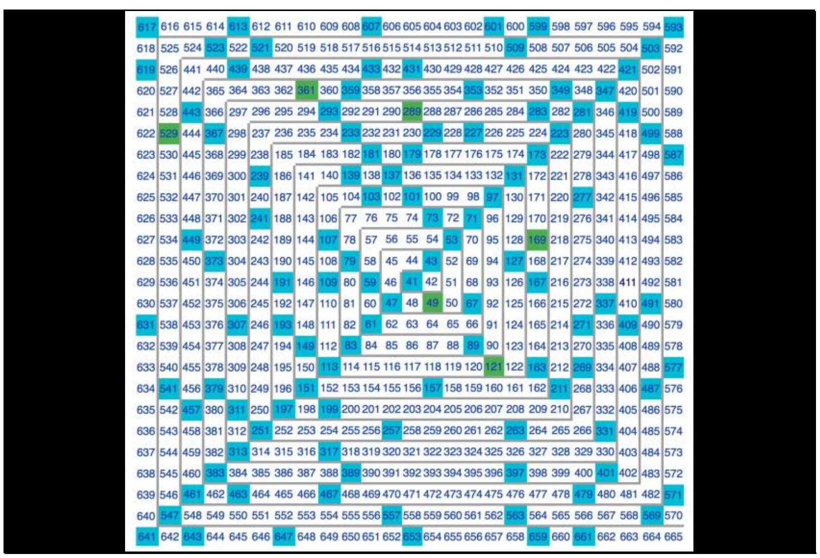
integers – numbers that have no fractional (or decimal) portion. The set of integers contains zero, the positive whole or natural numbers, and their additive inverses (negative numbers)

rational numbers – numbers that can expressed as the quotient (fraction) of two integers p/q with q≠0. This includes all the integers (which are divisible by 1), and all non-integers whose decimal expansion terminates after a fixed number of digits (e.g. 5/4 or 1.25), or begins to repeat a fixed sequence of digits (e.g. 4/3 or 1.33333…)

real numbers – a value of a continuous quantity that can express the distance along a line. Most real numbers are irrational.



Mathematicians have played all sorts of games with primes.
This one is the Ulam spiral invented by Stanislav Ulam in 1963. This spiral begins at 1 and rotates counterclockwise. What is fascinating is that primes are found on the diagonals (red) and squares also line up (blue)



This shows the Ulam spiral beginning at 41, with the primes shown in blue (and squares of primes in green).

Euler showed that n2 +n + 41 gives primes from for n between 0 and 39. However, the formula has not been generalized.
This is related to the blue diagonal in Ulam's spiral.

The best number from The Big Bang:

The best number is **73**. Why? **73** is the 21st prime number. Its mirror, **37**, is the 12th and its mirror, **21**, is the product of multiplying **7** and 3... and in binary **73**is a palindrome, **1001001**, which backwards is **1001001**.

https://www.youtube.com/watch?v=TIYMmbHik08

Magic Squares

Albrecht Dürer's
Magic Square

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Melencolia I, 1514
Albrecht Dürer

The philosophers of the Middle Ages were enamored of numbers. In the same way that the alchemists sought the philosopher’s stone, the numerologists sought out magic combinations. In a magic square the sum of the numbers along any horizontal or vertical line and along either diagonal is the same – in the case of Dürer’s square this value is 34.

From the description on the Metropolitan Museum of Art website:

Melencolia I is a depiction of the intellectual situation of the artist and is thus, by extension, a spiritual self-portrait of Dürer. In medieval philosophy each individual was thought to be dominated by one of the four humors; melancholy, associated with black gall, was the least desirable of the four, and melancholics were considered the most likely to succumb to insanity. Renaissance thought, however, also linked melancholy with creative genius; thus, at the same time that this idea changed the status of this humor, it made the self-conscious artist aware that his gift came with terrible risks.

The winged personification of Melancholy, seated dejectedly with her head resting on her hand, holds a caliper and is surrounded by other tools associated with geometry, the one of the seven liberal arts that underlies artistic creation--and the one through which Dürer, probably more than most artists, hoped to approach perfection in his own work. An influential treatise, the *De Occulta Philosophia* of Cornelius Agrippa of Nettesheim, almost certainly known to Dürer, probably holds the explanation for the number I in the title: creativity in the arts was the realm of the imagination, considered the first and lowest in the hierarchy of the three categories of genius. The next was the realm of reason, and the highest the realm of spirit. It is ironic that this image of the artist paralyzed and powerless exemplifies Dürer's own artistic power at its superlative height.

https://www.metmuseum.org/art/collection/search/336228

Fermat's Last Theorem (1637)

This conjecture was written in the margins of a copy of Diophantus' *Arithmetica*.

The equation $a^n + b^n = c^n$ has no solution for any integer value of n greater than 2.

Andrew Wiles published the first proof in 1994.

Pierre de Fermat (1607-1665)

Fermat's Little Theorem (1640)

For any prime p and any integer a, $a^p - a$ is a multiple of p.

$$a^p = a \ (\mathrm{mod}(p))$$

Examples: if a = 3 and p = 5, a^p = 243, 243-3 = 220 = 5 X 44
if a = 8 and p = 3, a^p = 512, 512-8 = 504 = 3 X 168

Euler published a proof in 1836.

Proving Fermat's Last Theorem was the goal of many a number theorist. The final proof by Andrew Wiles was 129 pages long.

In the last example one can determine that the number 504 is divisible by 3 since the sum of its digits (5+0+4 = 9) is divisible by three (another number theorem).

Still from the 2015 movie *The Man Who Knew Infinity*

This slide illustrates an anecdote about the great number theorist Srinavasan Ramanujan (18887-1920). This self-taught Indian mathematician came to Cambridge in 1914 to work with George Hardy. He died in 1920 as a result of a chronic disease – perhaps tuberculosis, perhaps amoebiasis. He proposed and proved many new theorems concerning numbers.

Hardy visited him in London during his illness:

> I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

The two different ways are:
$1729 = 1^3 + 12^3 = 9^3 + 10^3$

Number theory is the most abstract of the sciences. Carl Friedrich Gauss (1777-1855) said, "Mathematics is the queen of the sciences - and number theory is the queen of mathematics.“ For years it has fascinated us but has not had much in the way of application. It was pure thought. Leonard Dickson remarked "Thank God that number theory is unsullied by any application.”
This has changed. Nowadays, number theory is applied extensively in cryptography and computer programming.

Portrait by Frans Hals, 1648

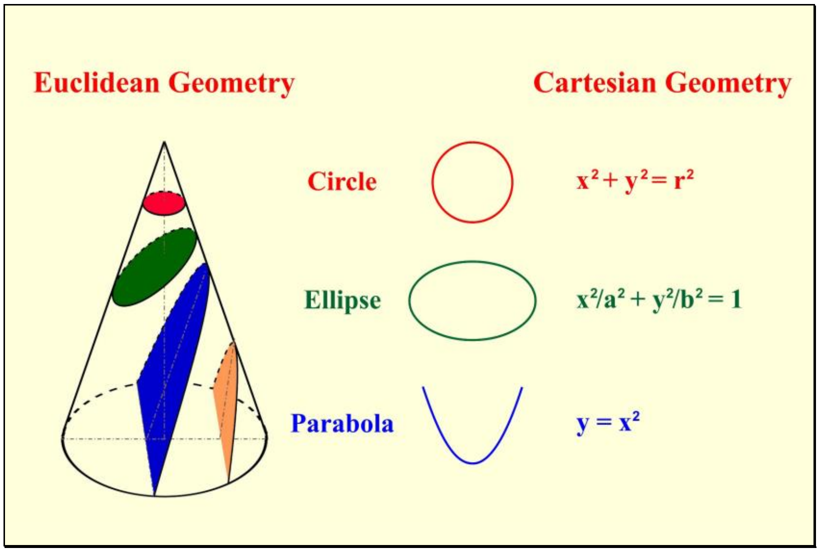
René Descartes (1596-1650)

Born near Tours, Descartes was trained in law at the University of Poitiers. After serving in the army he settled in the Netherlands in 1628. He where he wrote various essays on science but was cautious about publishing these because of the Galileo trial in 1633. In 1637 Descartes published the essays on meteors, optics, and geometry. The work on geometry established the field of analytic geometry. He later turned to philosophy and proposed his “Cogito ergo sum” (1641).

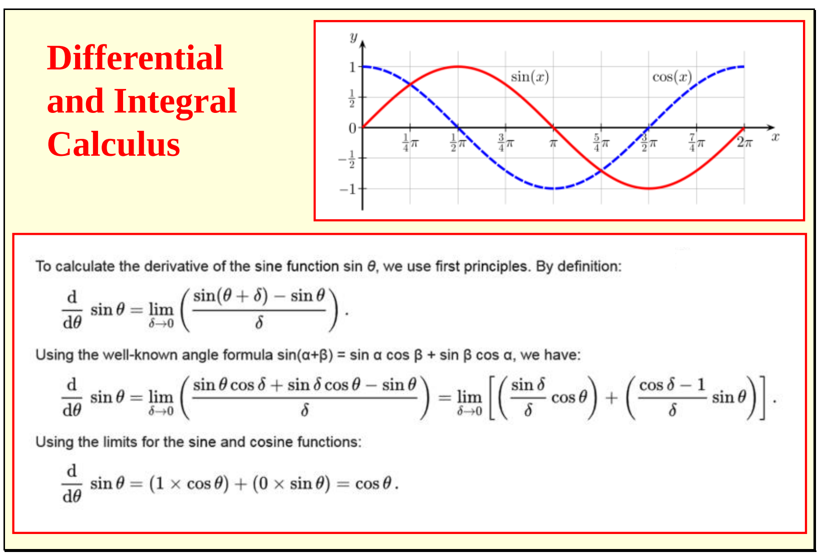
Pierre de Fermat (1607-1665) worked on analytic geometry at the same time and Descartes was aware of his work. Fermat called it the “method of loci.”

In 1649 Descartes was invited to the court of Queen Cristina of Sweden. Descartes went to Stockholm that winter. He and Queen Cristina did not get along, and the northern winter was very cold. Descartes caught pneumonia and died in February 1650.

Descartes did his best thinking when he was warm. In his 1641 *Meditations on First Philosophy* he describes how the method of skepticism – to doubt everything until one is left with what cannot be doubted – came to him while he was sitting warmly by the fireplace.

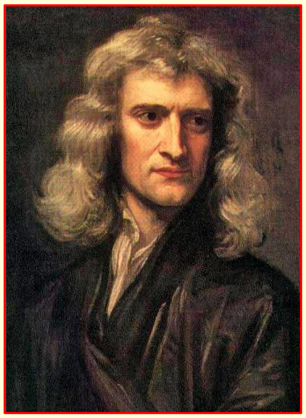


The yellow curve is the hyperbola with the equation $y^2/a^2 – x^2/b^2 = 1$



So now we turn to the development of calculus in the 17th Century:

Calculating the derivative of a sine function is taken from
https://en.wikipedia.org/wiki/Differentiation_of_trigonometric_functions



Portrait of Isaac Newton (1643-1727) by Godfrey Kneller, 1689

Portrait of Gottfried Wilhelm Leibniz (1646-1715) by Andreas Scheits, 1710

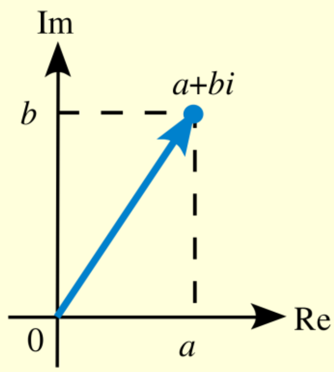
Newton developed a theory of differential calculus in 1671 in a book entitled *Method of Fluxions*. However this book was not formally published until 1736 (after Newton's death). He used a geometric version of "infinitesimal calculus" in his *Principia Mathematica* of 1687.

Leibniz published his *New Method for Maximum and Minimum* in 1694. Despite claims by Newton and his colleagues that Leibniz had plagiarized the work of Newton, he appears to have discovered these procedures independently.

The current notation used in calculus – d(f(x)/dx for differentiation and ∫ f(x)dx for integration – come from Leibniz. Newton used a dot above the function for differentiation and a square before it for integration.

The acrimonious dispute between the supporters of the two mathematicians is discussed in https://en.wikipedia.org/wiki/Leibniz–Newton_calculus_controversy

Complex Numbers



First discussed by the Italian mathematician Rafael Bombelli (1526-1572), complex number composed of a **Re**al and an **Im**aginary part, allow us to consider measurements in a 2-dimensional plane.

Leonhard Euler (1707-1783), a Swiss mathematician, used these complex numbers extensively in his calculations, and proposed the following identity:

$$e^{i\pi}+1=0$$

where **e** is the base of the natural logarithms, ***i*** is the imaginary unit such that $i^2 = -1$ and π the ratio of a circle's circumference to its diameter.

Complex numbers became very important is the calculation of planetary orbits and relationships.

Logarithms were invented by The English mathematician John Napier (1550-1617). A logarithm of a given number is the exponent (or power) to which a base must be raised to produce that number. For example the logarithm of 2 using base 10 is 0.30103. When 10 is raised to the power 0.30103 it equals 2.

e= 2.718281828

Both e and π are transcendent numbers. They cannot be obtained as the root of a non-zero polynomial equation with integer coefficients.

Euler's identity is considered as the most beautiful of all mathematical formulae, far more powerful than any medieval magic square or alchemical incantation.

If you want something to keep the demons away – this is it!

Probability and Statistics

Statistics is the science that deals with the collection and analysis of numerical data. Probability is the likelihood that an event will occur. Statistics uses probability distributions to assess sets of data.

The concepts of probability were used to decipher coded messages by Al-Kindi (801-873 CE) in Baghdad. Blaise Pascal (1623-1662) and Pierre-Simon Laplace (1749-1827) began the formal mathematical study of probability. Gauss used probability to analyze data and predict the orbits of planets.

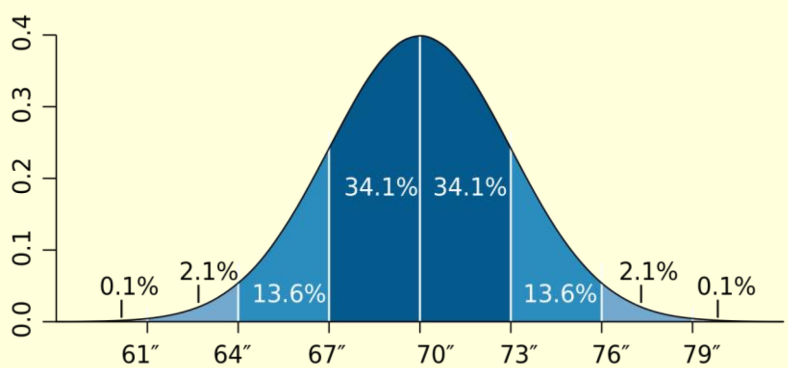
Carl Friedrich Gauss (1777-1855)

Gauss' portrait, painted by Christian Albrecht Jensen in 1840, shows him with his thinking cap on. This is actually part of his academic robe. However, many scientists and philosophers of that period seem to wear caps of some sort in their portraits.

An example of probability: the 3-side has a 1 in 6 chance of being uppermost in the cast of an unbiased 6-sided die.
An example of statistics : the 3-side occurs 30 times in 100 casts of a die. This suggests that the die is not unbiased.

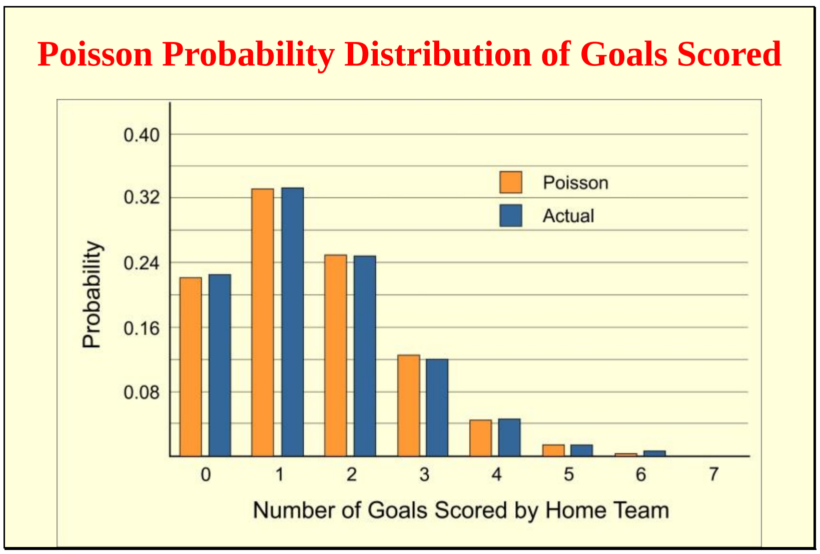
With the scientific revolution, measurements of the real world replaced abstract propositions. Induction replaced deduction. Statistics became important when inferring laws from empirical data. No measurement is completely accurate. There is always some noise (or variance) that must be considered when determining the limits of measurements or deciding that one measurement differs from another.

Normal (Gaussian) Distribution of Height
US Males, aged 18-86, 2002



This distribution is also known as the Bell Curve. It describes the distribution of measurements when the measurement is determined by multiple factors. The distribution is characterized by a mean (average) value and a standard deviation. 68% of measurements fall within ±1 standard deviations of the mean.

We usually start to suspect that something is wrong when 5 heads occur in a row. The probability of this occurring is 0.5^5 or 0.03125.
Generally we consider something that occurs with a probability of less than 0.05 as significant – as an unusual outlier. In the normal distribution this occurs when the measurement is more than 2 standard deviations beyond the mean.



The Poisson distribution is named after Siméon Denis Poisson (1781-1840).

The distribution gives the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant rate.
The number of goals scored in a professional soccer match fits this type of distribution. The average number of goals scored is just over 1 but occasionally the number can reach 7. Any one betting on the Premier League needs to know their Poisson distribution!

As the average number of events increases (e.g. the points scored in a basketball game) the distribution approaches the Gaussian distribution.

Bayes' Theorem

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

This theorem was initially proposed by Thomas Bayes (1701-1761), a British Clergyman and further developed by Pierre-Simon Laplace (1749-1827). It helps us to change our estimates of what will happen based on new evidence.

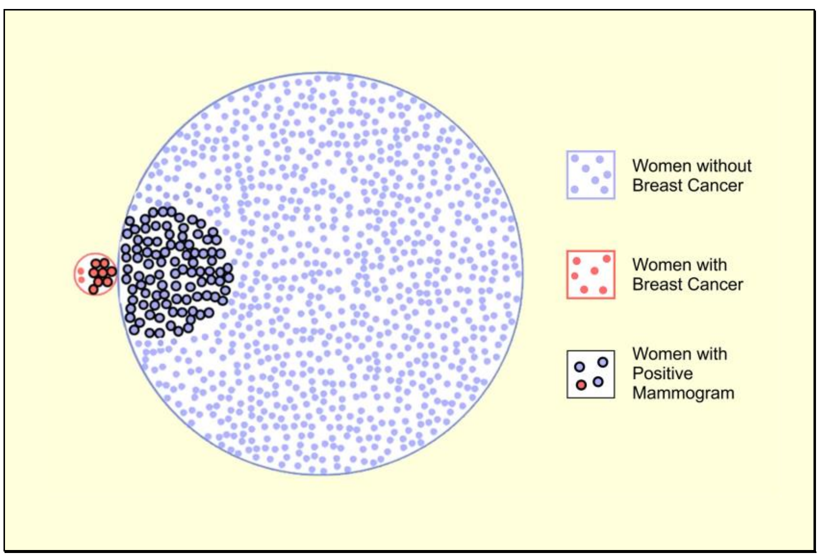
A woman at age 40 has a low probability (1%) of having breast cancer – P(A). The mammography test is positive in 80% of patients with cancer – P(B|A). However, the test gives a false positive result in 9.6% of women without cancer, and the overall probability of a positive test is 10.5% – P(B). If the woman's mammogram comes back positive, what are her chances of having cancer?

less than 20% 20-60% more than 60%

The mammography example is worked out more fully in the video
https://www.youtube.com/watch?v=D8VZqxcu0I0

According to Bayes' Theorem , the probability is 0.80 X 0.01 / 0.105 which is only 7.6%.

This is counter-intuitive – most people would assess the chances as much higher. Few people even physicians are aware of Bayesian probabilities



The explanation of the results is illustrated in this slide which shows the relative probabilities of the events. There are very few patients with cancer (pink). Although many of these will give a positive mammogram test (pink with black outline), many more normal patients will also test positively (blue with black outline).

Boolean Algebra

George Boole (1815-1864), a self-taught English mathematician, published in 1854 a book entitled *The Laws of Thought*. This considered the ideas of implication and whether propositions are true or false. He used algebraic symbols but these were confusing. A better set of logical symbols was developed by Alfred North Whitehead and Bertrand Russell in *Principia Mathematica* (1910-1913).

modus ponens
(method of affirming)

$$((p \rightarrow q) \wedge p) \vdash q$$

modus tollens
(denying the consequent)

$$(p \rightarrow q) \vdash (\sim q \rightarrow \sim p)$$

$$((p \rightarrow q) \wedge \sim q) \vdash \sim p$$

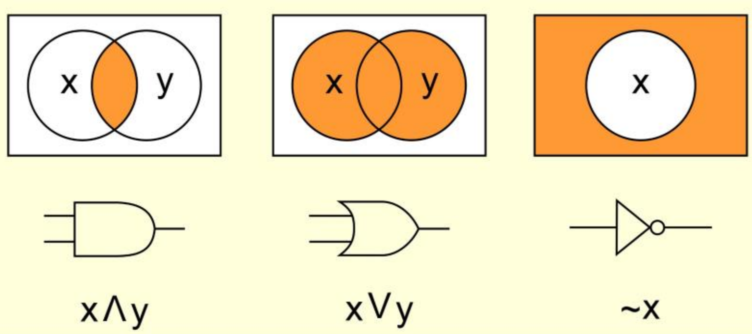
The “translation” of the symbolic expressions

If p implies q and p then q.

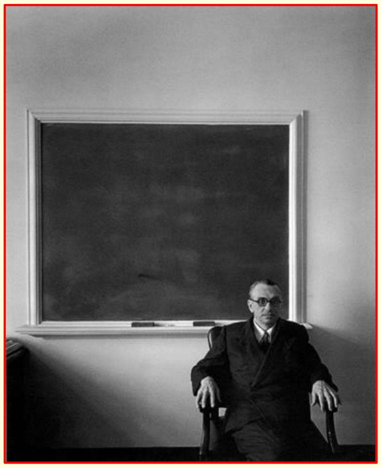
If p implies q then when q is not true neither is p
If p implies q and q is not true then neither is p

Logical Symbols

The following shows Venn-diagram representations and logical circuit symbols for conjunction (AND), disjunction (OR) and negation (NOT):



Boolean logic became the basis of electronic computation.



Portrait of Kurt Gödel (1906-1978) by Arnold Newman, 1956

Incompleteness

Alfred North Whitehead and Bertrand Russell in their *Principia Mathematica* (1910-3) attempted to show how all of mathematics could be derived logically from a limited set of axioms and rules.

In 1931 Kurt Gödel, an Austrian logician published a paper proving that this was impossible. In the paraphrase of Douglas Hofstadter in his 1979 book *Gödel, Escher Bach: An Eternal Golden Braid*:

All consistent axiomatic formulations of number theory include undecidable propositions

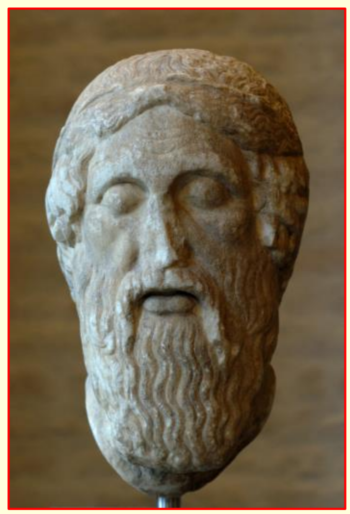
Epimenides of Knossos

Epimenides was a semi-mythical philosopher-poet who lived in Crete in the 7th or 6th Century BCE.

He proposed the paradoxical statement:

All Cretans are liars.

If the statement is true, then the Cretan Epimenides is lying and the statement is false.



We come to the end of the presentation by returning to the ancient world – Epimenides of Knossos. The Epimenides paradox is often used as an example of Gödel's incompleteness.

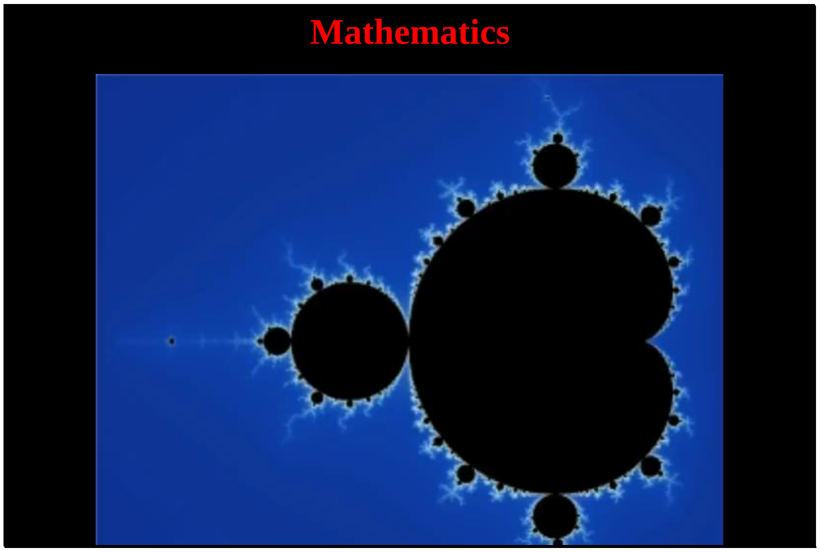
However, the Epimenides paradox can actually be solved. If the statement is false then some Cretans are honest. This does not include Epimenides.

Gödel's undecidability is better illustrated by the simple self-contradicting statement:

This statement is false.

If the statement is true then the statement is false, and vice versa.

The head in the Munich Glyptpthek was long thought to represent Epimenides but may more likely represent Homer.



https://en.wikipedia.org/wiki/Mandelbrot_set#/media/File:Mandelbrot_sequence_new.gif

The program examines the limits of a set of numbers plotted in 2-dimensional space at various levels of magnification. The result is "fractal" in that it is "infinitely self-similar." The Mandelbrot set was first investigated by Adrien Douady and John Hubbard in 1985 and named in honor of Benoit Mandelbrot, a Polish mathematician, who coined the term "fractal" (fragmented) in the 1960s to describe data sets that displayed self-similarity.

You are probably expecting me to explain this. I cannot! Teachers do not know everything.

If you wish to have some explanation of the Mandelbrot set try https://youtube.com/watch?v=NGMRB4O922I&feature=youtu.be